Project Dynamical Systems

6 April 2018

university of groningen

Exam Problem Sheet

The exam consists of 5 problems. You have 120 minutes to answer the questions. Give brief but precise answers. You can achieve 50 points in total which includes a bonus of 5 points.

1. [**3+3+3 Points.**]

   For each of the following bifurcations, plot the bifurcation diagram, describe in words the bifurcation scenario, and give explicit examples of a time continuous system showing the respective bifurcation.

   (a) Saddle node bifurcation.

   (b) Transcritical bifurcation.

   (c) Pitchfork bifurcation.

2. [**3+1+3+1 Points.**]

   Consider the differential equation $x' = x + \cos t$.

   (a) Find the general solution of this equation.

   (b) Prove that there is a unique periodic solution for this equation.

   (c) Compute the Poincaré map $p : t = 0 \mapsto t = 2\pi$ for this equation and use this to verify again that there is a unique periodic solution.

   (d) Use part (c) to determine the stability of the periodic solution from the Poincaré map.

3. [**1+4+1+2+3 Points.**]

   The differential equation

   $$x'' + bx' + kx = 0 \,,$$

   describes the motion of a particle (with mass $m = 1$) on which a spring force is acting (a so called *harmonic oscillator*). The constants satisfy $b \geq 0$ (b is the friction constant) and $k > 0$ (k is the spring constant).

   (a) Show that the differential equation is equivalent to the planar system

   $$X' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} X \,.$$

   (b) Identify the regions in the relevant portion of the $b - k$ plane where the system above has similar phase portraits. In each case sketch the phase portrait and describe the motions in words.

– **please turn over** –

(c) Show from the eigenvalues that for $b > 0$, the equilibrium at the origin is asymptotically stable.

(d) Show that $L(x, y) = \frac{1}{2}y^2 + \frac{1}{2}kx^2$ is a Lyapunov function for the system.

(e) State Lasalle's Invariance Principle.

(f) Use Lasalle's Invariance Principle to prove again that the equilibrium at the origin is asymptotically stable for $b > 0$ and that in this case the basin of attraction is the full plane. (Hint: consider disks of arbitrary radius centered at the origin.)

4. **[8 Points.]**

Consider the family of one-dimensional systems $x' = f(x, a)$ with parameter $a \in \mathbb{R}$. Suppose that for $x_0, a_0 \in \mathbb{R}$,

(i) $f(x_0, a_0) = 0$,

(ii) $\frac{\partial f}{\partial x}(x_0, a_0) = 0$,

(iii) $\frac{\partial^2 f}{\partial x^2}(x_0, a_0) \neq 0$, and

(iv) $\frac{\partial f}{\partial a}(x_0, a_0) \neq 0$.

Show that the system has a saddle-node bifurcation at (x_0, a_0).

5. **[3+6 Points.]**

(a) State the precise definition of chaos for a discrete time system $x_{n+1} = f(x_n)$, $n \in \mathbb{Z}_{\geq 0}$, given by a map $f : I \to I$ where I is a compact interval in $\mathbb{R}$.

(b) Argue that the doubling map

$$f : [0, 1] \to [0, 1], \quad x \mapsto 2x \bmod 1$$

is chaotic.

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